Close Wed: HW_2A, 2B, 2C (5.3,5.4,5.5)

5.5 The Substitution Rule

Entry Task (Motivation):

1. Find the following derivatives

Function	Derivative?
$\cos(x^2)$	$-\sin(x^2)\ 2x$
$\sin(x^4)$	$\cos(x^4)\ 4x^3$
$e^{\tan(x)}$	$e^{\tan(x)}\ \sec^2(x)$
$(\ln(x))^3$	$3(\ln(x))^2\ \frac{1}{x}$
$\ln(x^4+1)$	$\frac{1}{x^4+1}\ 4x^3$

2. Rewrite as integrals:

$$\int -\sin(x^2)\ \underline{2x\ dx} = \cos(x^2) + C$$

$$\int \cos(x^4)\ \underline{4x^3\ dx} = \sin(x^4) + C$$

$$\int e^{\tan(x)}\ \underline{\sec^2(x)\ dx} = e^{\tan(x)} + C$$

$$\int 3(\ln(x))^2\ \underline{\frac{1}{x}\ dx} = (\ln(x))^3 + C$$

$$\int \frac{1}{x^4+1}\ \underline{4x^3\ dx} = \ln(x^4+1) + C$$

3. Guess and check the answer to:

$$\int 7x^6 \sin(x^7)\, dx = \int \sin(u)\,du$$

$$= -\cos(u) + C$$

$$= \boxed{-\cos(x^7) + C}$$

$u = x^7$

$\frac{du}{dx} = 7x^6 \Leftrightarrow du = 7x^6 dx$

Observations:

1. We are reversing the "chain rule".
2. In each case, we see
   "inside" = a function inside another
   "outside" = derivative of inside

To help us mechanically see these connections, we use what we call:

The Substitution Rule:
If we write u = g(x) and du = g'(x) dx, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Ex $\int e^{\tan(x)} \sec^2(x)dx$ $\quad u = \tan(x)$, $\quad du = \sec^2(x)dx$

$= \int e^u \, du$

$= e^u + C = \boxed{e^{\tan(x)} + C}$

Aside (you do not need to write this)

Some theory

Recall:

$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

If we replace u = g(x), then we are "transforming" the problem from one involving *x* and *y* to one with *u* and *y*.

This changes ***everything*** in the set up. The lower bound, upper bound, widths, and integrand all change!

Recall from Math 124 that

$$g'(x) = \frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$$

(with more accuracy when Δx is small)

Thus, we can say that

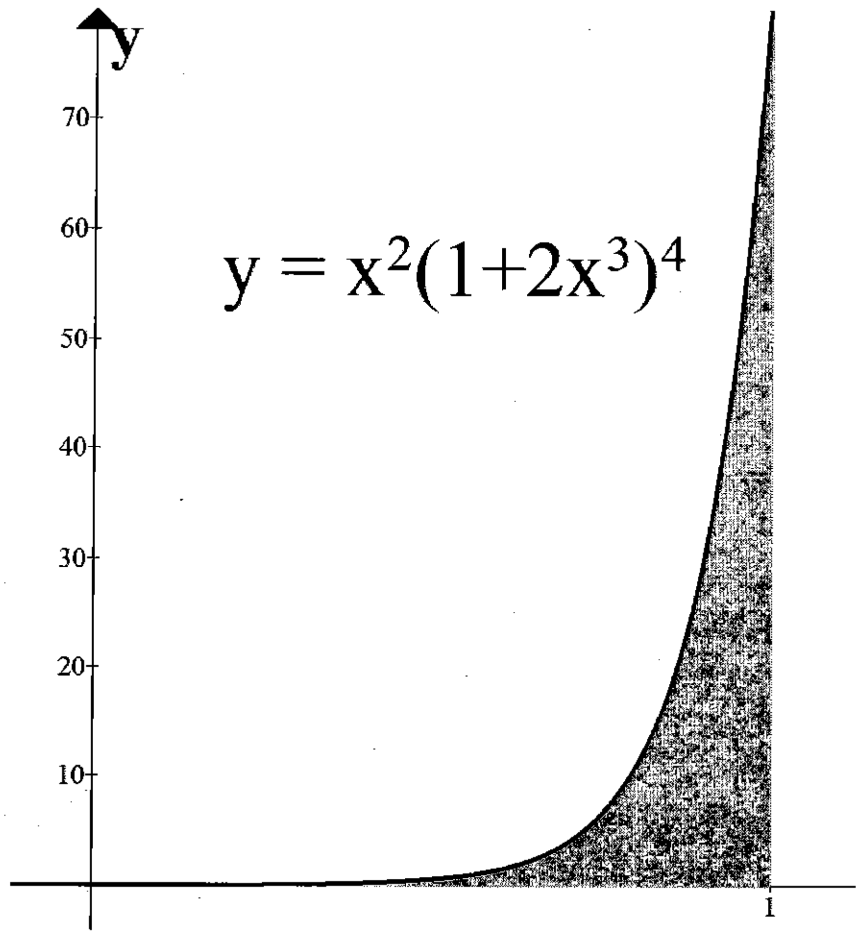
$$g'(x)\Delta x \approx \Delta u$$

In other words, if the width of the rectangles using *x* and *y* is Δx, then the width of the rectangles using *u* and *y* is $g'(x)\Delta x$.

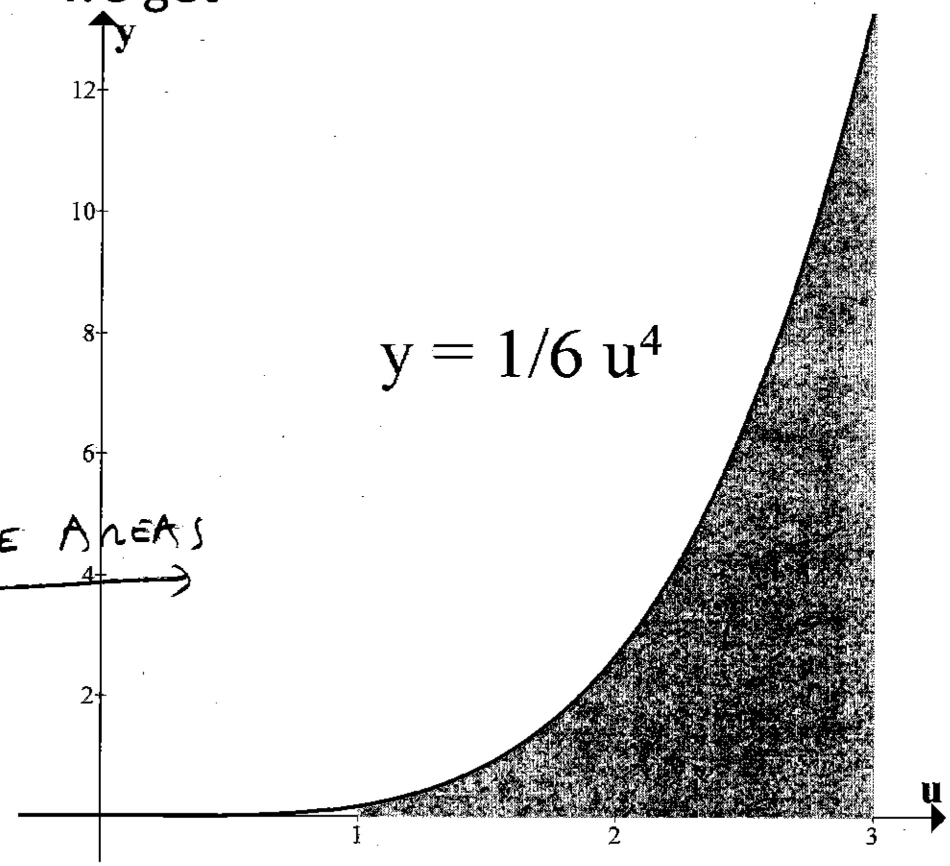
And if we write $u_i = g(x_i)$, then

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\Delta u\\ &= \int_{g(a)}^{g(b)} f(u)du\end{aligned}$$

Here is a visual example of this transformation



Using u = 1 + 2x³ and du = 6x² dx, we get



SAME AREAS

$$\int_0^1 x^2(1+2x^3)^4 dx$$

$u = 1 + 2x^3$

$du = 6x^2 dx$

$\frac{1}{6x^2} du = dx$

$$\int_1^3 \frac{1}{6} u^4 du$$

$$\int_1^3 \cancel{x^2}\, u^4 \frac{1}{6\cancel{x^2}} du$$

Examples:

First, try u = “inside function”

1. $\int x^4(1+x^5)^{31}\,dx$

$u = 1 + x^5$

$du = 5x^4\,dx$

$\frac{1}{5x^4}du = dx$

$= \int \cancel{x^4}\, u^{31} \frac{1}{5\cancel{x^4}}\,du$

$= \frac{1}{5}\int u^{31}\,du$

$= \frac{1}{5}\,\frac{1}{32}u^{32} + C$

$\boxed{\frac{1}{160}(1+x^5)^{32} + C}$

2. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$= \int \frac{\sin(u)}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du$

$= 2\int \sin(u)\,du$

$= -2\cos(u) + C$

$= \boxed{-2\cos(\sqrt{x}) + C}$

3. $\int_2^3 x^2 e^{x^3}\,dx$

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3x^2}\,du = dx$

$= \int_8^{27} x^2 \; e^u \; \frac{1}{3x^2}\,du$

$= \frac{1}{3}\int_8^{27} e^u\,du$

$= \frac{1}{3}\left(e^u \Big|_8^{27} \right)$

$= \boxed{\frac{1}{3}\left(e^{27} - e^{8}\right)}$

4. $\int \frac{x \sin(x^2)}{\cos^2(\cos(x^2))}\,dx$

$u = \cos(x^2)$

$du = -2x\sin(x^2)\,dx$

$\frac{1}{-2x\sin(x^2)}\,du = dx$

$= \int \frac{\cancel{x\sin(x^2)}}{\cos^2(u)} \; \frac{1}{\cancel{-2x\sin(x^2)}}\,du$

$= -\frac{1}{2}\int \frac{1}{\cos^2(u)}\,du$

$= -\frac{1}{2}\int \sec^2(u)\,du$

$= -\frac{1}{2}\tan(u) + C$

$= \boxed{-\frac{1}{2}\tan(\cos(x^2)) + C}$

Examples:

Then, try u = "denominator"

1. $\int_0^1 \frac{x}{x^2+3}\,dx$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$= \int_3^4 \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$

$= \frac{1}{2}\int_3^4 \frac{1}{u}du$

$= \frac{1}{2}\ln|u|\Big|_3^4$

$= \frac{1}{2}(\ln(4)-\ln(3))$

$= \frac{1}{2}\ln(4/3)$

2. $\int \tan(x)\,dx$

$= \int \frac{\sin(x)}{\cos(x)}dx$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$\frac{1}{-\sin(x)}du = dx$

$= \int \frac{\sin(x)}{u}\frac{1}{-\sin(x)}du$

$= -\int \frac{1}{u}du$

$= -\ln|u| + C$

$= -\ln|\cos(x)| + C$

$= \ln|(\cos(x))^{-1}| + C$

$= \ln\left|\frac{1}{\cos(x)}\right| + C$

$= \boxed{\ln|\sec(x)| + C = \int \tan(x)\,dx}$

ADD TO TABLE OF KNOWN FACTS!

What to do when the "old" variable remains:

Examples:

1. $\int x^3\sqrt{2+x^2}\,dx$

$u = 2 + x^2 \rightarrow x^2 = u - 2$

$du = 2x\,dx$

$\frac{1}{2x}\,du = dx$

$\int x^{\cancel{3}}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du$

$= \frac{1}{2}\int (u-2)\sqrt{u}\,du$

$= \frac{1}{2}\int u^{3/2} - 2u^{1/2}\,du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right) + c$

$= \boxed{\frac{1}{5}(2+x^2)^{5/2} - \frac{2}{3}(2+x^2)^{3/2} + c}$

2. $\int \frac{x^7}{x^4+1}\,dx$

$x^4 = u - 1$

$u = x^4 + 1$

$du = 4x^3\,dx$

$\frac{1}{4x^3}\,du = dx$

$= \int \frac{x^{\cancel{7}\,4}}{u}\,\frac{1}{4\cancel{x^3}}\,du$

$= \frac{1}{4}\int \frac{u-1}{u}\,du$

$= \frac{1}{4}\int 1 - \frac{1}{u}\,du$

$= \frac{1}{4}(u - \ln|u|) + c$

$= \boxed{\frac{1}{4}(x^4+1) - \frac{1}{4}\ln|x^4+1| + c}$

Basic Integration Quiz Sheet

The following pages of integrals all can be evaluated by either simplification or u-substitution. The first 2 pages contain indefinite integrals. The last page contains definite integrals. By the end of the third week of class you should be able to complete the first 2 pages in 15-20 minutes and the last page in 15-20 minutes. So you should be able to complete these types of integral problems in about 1 minute or less each.

Note that our current methods are limited to these types of problems, there are lots of integrals we still are unable to do. (This means, on the first exam I can only ask you to evaluate integrals that can be completed using simplification or u-substitution.)

Evaluate all the following:

1. $\displaystyle\int 3x^{10} - \frac{\sqrt{x}}{x^2} + 4\,dx$

2. $\displaystyle\int dx$

3. $\displaystyle\int \sin(\tan(x))\sec^2(x)\,dx$

4. $\displaystyle\int x^7(1+x^8)^{31}\,dx$

5. $\displaystyle\int \tan(x) + \frac{\sin(x)}{\cos^2(x)} + 13xe^{x^2}\,dx$

6. $\displaystyle\int (5x^4 - 6x)\sqrt{x^5 - 3x^2 + 1}\,dx$

7. $\int x(1+x)^5 \, dx$

8. $\int \frac{\sqrt{3x^3} + 4x^2 - 11x}{\sqrt{x}} \, dx$

9. $\int \frac{x^5}{\sqrt{1+x^3}} \, dx$

10. $\int \cos(x)\sin(x) \, dx$

11. $\int \sin(13x) \, dx$

12. $\int e^{7x} \, dx$

13. $\int \cos(\frac{1}{4}x) \, dx$

14. $\int \sin(-5x) \, dx$

15. $\int e^{101x} \, dx$

16. $\int \cos(2x) \, dx$

17. $\displaystyle\int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^2 \sin(x^3)\ dx$

18. $\displaystyle\int_0^{\frac{\pi}{2}} e^{-3\cos(x)} \sin(x)\ dx$

19. $\displaystyle\int_{-10}^{-3} e^{\frac{1}{10}x}\ dx$

20. $\displaystyle\int_{\pi/3}^{\pi/4} \sin(-\frac{7}{8}x)\ dx$

21. $\displaystyle\int_2^3 \frac{x}{\sqrt{4+x^2}}\ dx$

22. $\displaystyle\int_1^5 \frac{x^3 - 2x^2 + x^{5/2}}{x^{1/3}}\ dx$

23. $\displaystyle\int_2^{2e} \frac{3-x}{2x}\ dx$

24. $\displaystyle\int_1^e \frac{(\ln(x))^3}{x}\ dx$